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Level-spacing distribution of a singular billiard

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In this paper, we examine the level-spacing distribution $P(S)$ of the rectangular billiard with a single pointlike scatterer, which is known as *almost integrable*. It is shown that the observed $P(S)$ is a new type, which is quite different from the previous conclusion. Even in the strong-coupling limit, the Poisson-like behavior rather than Wigner-like behavior is seen for $S \gtrsim 1$, although the level repulsion still remains in the small-S region. The difference from the previous works is analyzed in detail.

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To our knowledge, there are no bounded, undriven quantum systems that exhibit chaos (exponential instability), even if their classical counterparts are chaotic [1]. However, they seem to behave differently concerning their eigenvalues and wave functions, depending on whether the classical orbits are regular or chaotic. Taking the level statistics of quantum systems as an example, there is a conjecture that the classically chaotic systems obey the Wigner statistics, whereas the regular systems obey the Poisson statistics. It is one of the general trends in studying quantum systems to examine their properties in conjunction with those of the corresponding classical counterparts [2–5].

Quantum aspects of so-called *almost integrable* systems [6], which are classically nonchaotic, have attracted a renewed attention in recent years. Many numerical and theoretical works have already been done on the almost integrable systems [7–23]. One of them is the rectangular billiard with a single pointlike scatterer [18–21]. In Ref. [20], Šeba and Życzkowski examined this system over a wide range of energy excitation and revealed some new aspects. One of their conclusions is that the level-spacing distribution $P(S)$ becomes closer to the Wigner distribution as the coupling strength increases, although the fine structure [such as the asymptotic (large-S) behavior] does not conform to the prediction of the Gaussian orthogonal ensemble (GOE). The appearance of a Wigner-like distribution in this classically nonchaotic billiard has led several authors to conjecture that this might represent the quantum signature of chaos generated by the probabilistic nature of the wave function [13,18,20]. In order to emphasize that such phenomena originate from quantum effects, they are called *wave chaos* in Refs. [18] and [20]. Wave chaos is distinguished from *quantum chaos* in the sense that the latter means the properties of quantum systems if (and only if) their classical counterparts are chaotic.

In Ref. [20] the Green's-function method is used to investigate the rectangular billiard with a single pointlike scatterer. While it does not formally include any approximation, the truncation of the basis is inevitable in the actual numerical calculation. One should be most careful to decide the range of the basis because of the singularity of the interaction under consideration. In this Rapid Communication, we will examine the effect of the restriction of the basis on $P(S)$. This might appear to be a merely technical problem in the numerical treatment, without any physical interest. As we will see later, however, this is not the case, and the strictest care to the basis is essential for this system. In fact, the main conclusion is that the level-spacing distribution $P(S)$ in the strong-coupling limit never becomes Wigner-like but belongs to a new class, which might be said to be *intermediate* between regularity and chaos.

As the mathematical formalism is explained in full in Refs. [20] and [21], we only summarize those points that are necessary in the following discussion. The Hamiltonian of the rectangular billiard with a pointlike scatterer is formally given by

$$H = -\frac{\Delta}{2M} + v_0 \delta(x - x_0)\delta(y - y_0) , \tag{1}$$

where M is the mass of a particle, and v_0 and (x_0, y_0) are the strength and position of the scatterer, respectively. The Green's function of this system is given by

$$G(x,y;x',y';z) = G_F^{(0)}(x,y;x',y';z) + G_F^{(0)}(x,y;x_0,y_0;z)\frac{v_0}{1 - G^{(0)}(x_0,y_0;x_0,y_0;z)v_0} \times G_F^{(0)}(x_0,y_0;x',y';z) . \tag{2}$$

Here, z is the energy variable, $G_F^{(0)}$ is the Green's function of the billiard without any scatterer, and $G^{(0)}(x_0,y_0;x_0,y_0;z)$ describes the propagation of the particle that begins to propagate at the pointlike scatterer and ends there. Clearly, the second term on the right-hand side in Eq. (2) means the multiple scattering caused by the pointlike scatterer. [Although the authors of Ref. [20] call $G^{(0)}(x_0,y_0;x_0,y_0;z)$ with opposite sign in Eq. (2) the meromorphic function $\xi(z)$, we just call it the Green's function in the following discussion.] From Eq. (2), we see that, in the Green's-function method, the eigenvalue problem is equivalent to solving the following transcendental equation:

$$G^{(0)}(x_0,y_0;x_0,y_0;z)=\frac{1}{v_0}\ . \tag{3}$$

If the scatterer is located at the center of the rectangle, which is the case that we will examine in this paper, the Green's function with the Dirichlet condition on the border of the billiard is given by

$$G^{(0)}(z)\equiv G^{(0)}(x_0,y_0;x_0,y_0;z) = \frac{4}{l_x l_y}\sum_{n_x,n_y=1}^{\infty}\left[\frac{1}{z-E^{(0)}_{2n_x-1,2n_y-1}} + \frac{E^{(0)}_{2n_x-1,2n_y-1}}{(E^{(0)}_{2n_x-1,2n_y-1})^2+1}\right], \tag{4}$$

$$E^{(0)}_{n_x,n_y}=\frac{\pi^2}{2M}\left[\left[\frac{n_x}{l_x}\right]^2+\left[\frac{n_y}{l_y}\right]^2\right]. \tag{5}$$

Here, $E^{(0)}_{n_x,n_y}$ is the eigenvalue of the billiard without any obstacle and l_x and l_y are the side lengths of the rectangle ($x_0=l_x/2$ and $y_0=l_y/2$). One should notice that, when the scatterer is placed at the center of the billiard, the scatterer affects only even-even parity states. The special feature in the case of singular interaction can be seen in the second term of the Green's function in Eq. (4), while the Green's function $G_F^{(0)}$ in the case without any obstacle in the billiard does not have the corresponding term. (The appearance of this term is closely related to the boundary condition around the scatterer. In order to determine its exact form, one needs the help of some theorems in the functional analysis [20,21]. See also Refs. [24] and [25].) One realizes that each of two terms in the Green's function $G^{(0)}(z)$ has logarithmic divergence when summed separately, although the sum of them leads to a finite value.

We examine the case in which $M=8\pi$, $l_x=\pi/3$, and $l_y=3/\pi$. In this particular parametrization, the average density of even-even parity states is equal to 1 according to Weyl's formula.

To see a general feature of the Green's function $G^{(0)}(z)$, the schematic graph of the Green's function is shown in Fig. 1. Here, the eigenvalues $\{E^{(0)}_{2n_x-1,2n_y-1}\}$ of even-even parity states in the unperturbed system are renamed in ascending order as $\{E_n^{(0)}\}$. One can easily see that each eigenvalue E_n of the perturbed system is isolated between two unperturbed energies and becomes larger as one increases the strength of the coupling. In the strong-coupling limit ($v_0=\infty$), the set of eigenvalues is just that of zeros of the Green's function.

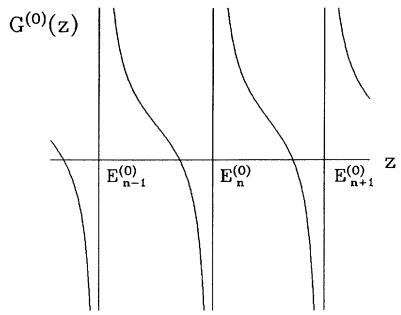

FIG. 1. Schematic graph of the Green's function in Eq. (4).

In order to get the solutions of Eq. (3) numerically, one must limit the range of the summation with minimum and maximum values of n, $n_{\min}$ and $n_{\max}$,

$$G^{(0)}_{\text{approx}}(z)=4\sum_{n=n_{\min}}^{n_{\max}}\left[\frac{1}{z-E_n^{(0)}}+\frac{E_n^{(0)}}{(E_n^{(0)})^2+1}\right]. \tag{6}$$

The prescription of the limitation in Ref. [20] is to take $n_{\min}=l-500$ and $n_{\max}=l+500$ for looking for a root E_l of Eq. (3) localized between $E_l^{(0)}$ and $E_{l+1}^{(0)}$. Hereafter, we refer to this prescription as the *truncation (I)* prescription. At first sight, this seems to be quite reasonable because the main contribution on the Green's function around the energy E_l comes from the terms that have n around l. According to the truncation (I), one gets the level-spacing distribution $P(S)$ without much numerical labor. As a typical example, we show the case of the strong-coupling limit in Fig. 2. This corresponds to Fig. 2(c) in Ref. [20] and of course shows quite similar structure. One might conclude from Fig. 2 that the level-spacing distribution of the rectangular billiard with a pointlike scatterer is almost Wigner-like in the strong-coupling limit.

We now examine the accuracy of truncation (I). Figure 3 shows the same calculation as above, except that

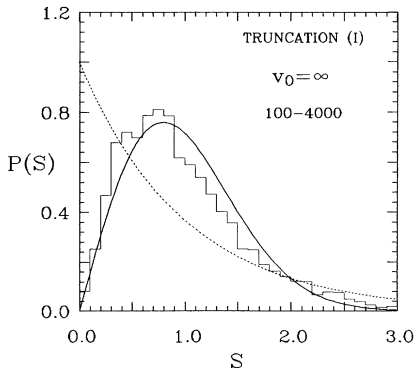

FIG. 2. Level-spacing distribution $P(S)$ in the case of the strong-coupling limit according to truncation (I); $n_{\min}=l-500$ and $n_{\max}=l+500$ in Eq. (6). Statistics are taken within the eigenstates indicated in the figure. The Wigner (solid line) and Poisson (broken line) distributions are also shown for reference.

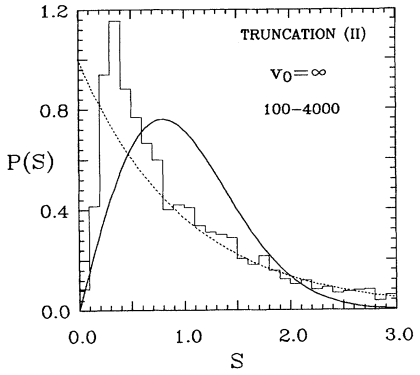

FIG. 3. Same as Fig. 2, except according to truncation (II); $n_{\min}=1$ and $n_{\max}=100\,000$ in Eq. (6).

$n_{\min}=1$ and $n_{\max}=100\,000$. Hereafter we refer to this case as the *truncation (II)* case. We will later justify this truncation of the basis in an analytic manner in this paper. One easily sees the drastic changes even in a qualitative level. For $S\gtrsim 1$, $P(S)$ is rather more Poisson-like than Wigner-like, although the level repulsion still remains in the small-S region. The level repulsion is regarded as a common feature among the various almost integrable systems [8,13,16,23]. For the almost integrable billiard with a single pointlike scatterer, it is rigorously proven in Ref. [21] that $S\le P(S)\le\frac{9}{4}S$ as $S\to 0$ and $P(S)\ge e^{-S}$ for large S hold. The level-spacing distribution in Fig. 3 is consistent with these estimates. Roughly speaking, one might say that it shows an *intermediate* feature between regularity and chaos.

In order to clarify the reason for the disagreement between the level-spacing distributions in Figs. 2 and 3, we estimate the numerical error in the Green's function related to the truncation of the basis. The numerical error comes from the terms that are neglected by the limitation of the summation in Eq. (6),

$$\delta G^{(0)}(z)\equiv G^{(0)}(z)-G^{(0)}_{\text{approx}}(z)$$

$$=4\left[\sum_{n=1}^{n_{\min}-1}+\sum_{n=n_{\max}+1}^{\infty}\right]\left[\frac{1}{z-E_n^{(0)}}+\frac{E_n^{(0)}}{(E_n^{(0)})^2+1}\right]. \quad (7)$$

To estimate the order of magnitude of the error, we consider the unperturbed energy $E_n^{(0)}$ as the continuous variable and replace the summation by the integral as

$$\delta G^{(0)}(z)\simeq 4\left[\int_0^{E_{n_{\min}}}+\int_{E_{n_{\max}}}^{\infty}\right]\left[\frac{1}{z-E}+\frac{E}{E^2+1}\right]dE\ . \quad (8)$$

Here, one should notice that the mean level density is constant and equal to 1 in our parametrization. The integral in Eq. (8) is elementary and leads to

$$\delta G^{(0)}(z)\simeq 4[F(z,E_{n_{\min}})-F(z,0)-F(z,E_{n_{\max}})]\ , \quad (9)$$

where the function F is defined by

$$F(z,E)=\tfrac{1}{2}\ln\frac{E^2+1}{(z-E)^2}\ . \quad (10)$$

If $1\ll E_{n_{\min}}<z<E_{n_{\max}}$ and $z\simeq(E_{n_{\min}}+E_{n_{\max}})/2$, then one obtains

$$\delta G^{(0)}(z)\simeq 4\left[\ln z+\ln\frac{E_{n_{\min}}}{E_{n_{\max}}}\right]. \quad (11)$$

This shows that if one evaluates, for example, E_{1000} ($\simeq 1000$) according to truncation (I), the numerical error

$$\delta G^{(0)}(1000)\simeq 4(\ln 1000+\ln\tfrac{500}{1500})\simeq 4(6.90-1.09)\simeq 23.2$$

is accompanied. Also, Eq. (11) shows that the error is much larger as one increases the energy. Clearly, the underestimation of the Green's function leads to the underestimation of the eigenvalues.

The accuracy of the zeros is not directly related to the magnitude of the error in the Green's function, but to the ratio between the magnitude of the error and the derivative of the Green's function at zero. Therefore, we further examine the derivative of the Green's function. As a typical example, we show in Table I some eigenvalues around E_{1000} obtained by truncation (I) and the derivative of the Green's function at the corresponding zero. For comparison, we show the result with truncation (II). It can be easily seen from Table I that, whereas the zero is fairly accurate if the derivative there is large enough compared to the error (about 20), this is not the case if the derivative is small. In fact, some eigenvalues have numerical errors comparable to the mean energy difference between nearest-neighbor levels. It is also unfortunate for truncation (I) that the sequence of the absolute value of the derivatives looks to be random. So, the accuracy of a zero just next to a very accurate one can be very poor. This of course has a serious influence on $P(S)$.

On the contrary, the numerical error by truncation (II) is given by

$$\delta G^{(0)}(z)\simeq -4F(z,E_{n_{\max}})\ , \quad (12)$$

and is quite small even for $E_{4000}\simeq 4000$,

$$\delta G^{(0)}(4000)\simeq -0.16\ .$$

Also, one can see that the large magnitude of the derivatives of the Green's function ensures the accuracy of the

TABLE I. Zeros and the derivatives of the Green's function at the corresponding zeros. The second and third columns show the results according to truncation (I), whereas the fourth and fifth columns show the case of truncation (II).

n	E_n	$\lvert(G^{(0)}_{\text{approx}})'(E_n)\rvert$	E_n	$\lvert(G^{(0)}_{\text{approx}})'(E_n)\rvert$
995	994.29	49	994.52	251
996	995.60	23	996.03	169
997	996.55	89	996.71	292
998	997.30	95	997.43	365
999	999.39	12	1000.13	110
1000	1000.46	782	1000.49	1032
1001	1001.95	16	1002.44	178
1002	1003.50	15	1004.24	111
1003	1004.75	77	1005.00	159
1004	1005.31	618	1005.34	873
1005	1006.32	30	1006.70	171

zeros. The absolute value of the typical error with a zero is estimated to be, at most, of the order of 10^{-3}, namely, 0.1% compared to the mean level spacing.

The physical reason why such a large phase space is necessary is obvious. It is the singularity of the interaction between the unperturbed levels. In fact, any two even-even parity states couple to each other with the same coupling strength when the scatterer point is located at the center of the billiard. The presence of singular points, which only have a negative curvature in the phase space, is a feature common to the almost integrable systems. Extreme care with numerical accuracy is required in order to analyze such systems.

The reminiscence of the Poisson-like behavior (regularity) for $S \gtrsim 1$ in the strong-coupling limit is surprising. The billiard with a pointlike scatterer seems to be in remarkable contrast to a polygonal billiard, at least concerning the level-spacing distribution [23]. For the latter system, the level-spacing distribution is considerably more Wigner-like rather than Poisson-like, even in the case of $g=2$ or 3, where g is the genus number for the polygonal billiard. (For the billiard under consideration, the genus number is $g=2$.) One possible interpretation might be that the degree of chaos of the quantized almost integrable billiard with pointlike scatterers is much smaller than previously believed. In order to draw a definite conclusion, a more extensive analysis is necessary.

In summary, we have shown that the level-spacing distribution $P(S)$ of the rectangular billiard with a pointlike scatterer in the strong-coupling limit belongs to a new class. Contrary to the previous conclusion, it does not show Wigner-like behavior, but shows Poisson-like behavior for $S \gtrsim 1$, although there remains the level repulsion in the small-S region. A wide range of the Fourier basis is demanded in order to get the correct eigenvalues of this system.

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[1] It has been proven by J. Ford and M. Ilg [Phys. Rev. A **45**, 6165 (1992)] that finite, bounded, and undriven quantum systems are not chaotic in the sense of chaos in the algorithmic complexity theory.

[2] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

[3] F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Berlin, 1991).

[4] L. E. Reichl, *The Transition to Chaos* (Springer-Verlag, New York, 1992).

[5] The study of the quantum behavior characteristic of systems whose classical motion exhibits chaos is called *quantum chaology* by M. V. Berry [Proc. R. Soc. London Ser. A **413**, 183 (1987)].

[6] In the literature, the term *pseudointegrable* is often used instead of the term *almost integrable,* which was first introduced by A. N. Zemlyakov and A. B. Katok in Ref. [7].

[7] A. N. Zemlyakov and A. B. Katok, Math. Notes **18**, 291 (1975).

[8] P. J. Richens and M. V. Berry, Physica D **2**, 495 (1981).

[9] F. S. Henyey and N. Pomphrey, Physica D **6**, 78 (1982).

[10] M. V. Berry and M. Wilkinson, Proc. R. Soc. London Ser. A **392**, 15 (1984).

[11] B. Eckhardt, J. Ford, and F. Vivaldi, Physica D **13**, 339 (1984).

[12] E. Gutkin, Physica D **19**, 311 (1986).

[13] T. Cheon and T. D. Cohen, Phys. Rev. Lett. **62**, 2769 (1989).

[14] B. Mirbach and H. J. Korsch, Nonlinearity **2**, 327 (1989).

[15] J. H. Hannay and R. J. McCraw, J. Phys. A **23**, 887 (1990).

[16] C. H. Lewenkopf, Phys. Rev. A **42**, 2431 (1990).

[17] D. Biswas and S. R. Jain, Phys. Rev. A **42**, 3170 (1990).

[18] P. Šeba, Phys. Rev. Lett. **64**, 1855 (1990).

[19] T. Cheon, T. Mizusaki, T. Shigehara, and N. Yoshinaga, Phys. Rev. A **44**, R809 (1991).

[20] P. Šeba and K. Życzkowski, Phys. Rev. A **44**, 3457 (1991).

[21] S. Albeverio and P. Šeba, J. Stat. Phys. **64**, 369 (1991).

[22] K. Życzkowski, Acta Phys. Pol. B **23**, 245 (1992).

[23] A. Shudo and Y. Shimizu, Phys. Rev. E **47**, 54 (1993).

[24] J. Zorbas, J. Math. Phys. **21**, 840 (1980).

[25] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, New York, 1988).